Topic 8 -
Undetermined Coefficients

We now learn a method to guess a particular solution y_p to

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

where a_2, a_1, a_0 are constants.

This method is called the method of undetermined coefficients.

It will only work for certain $b(x)$.

It will first require finding the homogeneous solution y_h to

$$a_2 y'' + a_1 y' + a_0 y = 0$$

There will be several cases to consider.

Ex: Find the general solution to

$$y'' + 3y' + 2y = 2x^2$$

Step 1: Solve the homogeneous equation

$$y'' + 3y' + 2y = 0$$

The characteristic polynomial is

$$r^2 + 3r + 2 = 0$$

$$(r+2)(r+1) = 0$$

$$r = -1, -2$$

The general homogeneous solution is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

Step 2: Guess what y_p is.

We have $b(x) = 2x^2$.

This $b(x)$ doesn't occur as part of y_h.

Note that $b(x)$ has derivatives

$$2x^2, \; 4x, \; 4$$

Thus we guess

$$y_p = Ax^2 + Bx + C$$

make y_p as combos of derivatives of $b(x)$

since $b(x)$ and it's derivatives are of these forms.

Now plug this y_p into $y'' + 3y' + 2y = 2x^2$.

We have

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

So, plugging y_p into $y'' + 3y' + 2y = 2x^2$ gives

$$(2A) + 3(2Ax + B) + 2(Ax^2 + Bx + C) = 2x^2$$

Rearrange

$$2A + 6Ax + 3B + 2Ax^2 + 2Bx + 2C = 2x^2$$

Group like terms and set to 0 to get

$$(2A + 3B + 2C) + (6A + 2B)x + (2A - 2)x^2 = 0$$

Thus we get

$$\boxed{\begin{aligned} 2A+3B+2C &= 0 \quad (1)\\ 6A+2B &= 0 \quad (2)\\ 2A-2 &= 0 \quad (3)\end{aligned}}$$

(3) gives $A=1$.

Plug $A=1$ into (2) to get $6+2B=0$.

So, $B=-3$.

Plug $A=1, B=-3$ into (1) to get $2-9+2C=0$

So, $C=7/2$.

Thus,

$$\begin{aligned} y_p &= Ax^2+Bx+C \\ &= x^2-3x+\frac{7}{2}\end{aligned}$$

solves

$$y''+3y'+2y=2x^2.$$

Step 3: Thus, the general solution to

$$y''+3y'+2y=2x^2$$

is

$$y=y_h+y_p=c_1e^{-x}+c_2e^{-2x}+x^2-3x+\frac{7}{2}$$

Ex: Let's find the general solution to

$$y'' - y' + y = 2\sin(3x)$$

Step 1: First we solve the homogeneous equation

$$y'' - y' + y = 0$$

which has characteristic equation

$$r^2 - r + 1 = 0$$

The roots are

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \sqrt{3}\, i$$

Thus, the general homogeneous solution is

$$y_h = c_1 e^{\frac{1}{2}x} \cos(\sqrt{3}x) + c_2 e^{\frac{1}{2}x} \sin(\sqrt{3}x)$$

Step 2: Now we guess a particular solution to

$$y'' - y' + y = \underbrace{2\sin(3x)}_{b(x)}$$

Note that $b(x) = 2\sin(3x)$
$$\left.\begin{aligned} b(x) &= 2\sin(3x) \\ b'(x) &= 6\cos(3x) \\ b''(x) &= -18\sin(3x) \\ &\vdots \end{aligned}\right\}$$
derivates contain $\sin(3x)$ and $\cos(3x)$

Thus we guess

$$y_p = A\sin(3x) + B\cos(3x)$$

Notice that none of the terms of y_p occur in y_h.

Let's try plugging y_p into $y'' - y' + y = 2\sin(3x)$.

We get

$$\begin{aligned} y_p &= A\sin(3x) + B\cos(3x) \\ y_p' &= 3A\cos(3x) - 3B\sin(3x) \\ y_p'' &= -9A\sin(3x) - 9B\cos(3x) \end{aligned}$$

Plugging into $y'' - y' + y = 2\sin(3x)$ gives

$$\left(-9A\sin(3x) - 9B\cos(3x)\right) - \left(3A\cos(3x) - 3B\sin(3x)\right) + \left(A\sin(3x) + B\cos(3x)\right) = 2\sin(3x)$$

This gives

$$\underbrace{(-3A-8B)}_{\text{must be } 0}\cos(3x)+\underbrace{(-8A+3B-2)}_{\text{must be } 0}\sin(3x)=0$$

Thus we get

$$\boxed{\begin{aligned} -3A-8B&=0 \quad ① \\ -8A+3B-2&=0 \quad ② \end{aligned}}$$

① gives $A=-\frac{8}{3}B$.

Plug into ② to get $-8\left(-\frac{8}{3}B\right)+3B-2=0$

So, $\frac{73}{3}B=2$.

Thus, $B=\frac{6}{73}$

So, $A=-\frac{8}{3}B=-\frac{8}{3}\left(\frac{6}{73}\right)=-\frac{16}{73}$

Thus,

$$y_p=\frac{-16}{73}\sin(3x)+\frac{6}{73}\cos(3x)$$

is a particular solution to

$$y''-y'+y=2\sin(3x)$$

Step 3: Thus, the general solution to

$$y'' - y' + y = 2\sin(3x)$$

is

$$y = y_h + y_p$$

$$= c_1 e^{\frac{1}{2}x}\cos(\sqrt{3}x) + c_2 e^{\frac{1}{2}x}\sin(\sqrt{3}x)$$

$$-\frac{16}{73}\sin(3x) + \frac{6}{73}\cos(3x)$$

Here is a table to help you make your guess for y_p for $a_2 y'' + a_1 y' + a_0 y = b(x)$

$b(x)$	y_p
constant	A
$5x - 3$	$Ax + B$
$10x^2 - x + 1$	$Ax^2 + Bx + C$
$x^3 - x + 10$	$Ax^3 + Bx^2 + Cx + D$
$\sin(6x)$	$A\cos(6x) + B\sin(6x)$
$\cos(6x)$	$A\cos(6x) + B\sin(6x)$
e^{3x}	Ae^{3x}
$(2x+1)e^{3x}$	$(Ax+B)e^{3x}$
$x^2 e^{3x}$	$(Ax^2+Bx+C)e^{3x}$
$e^{3x}\sin(4x)$	$Ae^{3x}\cos(4x) + Be^{3x}\sin(4x)$
$5x^2\sin(4x)$	$(Ax^2+Bx+C)\cos(4x)$ $+ (Dx^2+Ex+F)\sin(4x)$
$xe^{3x}\cos(4x)$	$(Ax+B)e^{3x}\cos(4x)$ $+ (Cx+D)e^{3x}\sin(4x)$

What if in

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

you have multiple terms in $b(x)$.

That is, what if

$$b(x) = b_1(x) + b_2(x) + \cdots + b_n(x).$$

Then you guess a term for each $b_i(x)$ and add your guesses together.

Ex: Let's solve

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$$

Step 1:

First solve the homogeneous equation

$$y'' - 2y' - 3y = 0$$

The characteristic equation is

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0$$

Thus, the general homogeneous solution is

$$y_h = c_1 e^{3x} + c_2 e^{-x}$$

step 2: Now we guess a particular solution for

$$y'' - 2y' - 3y = \underbrace{(4x-5)}_{\text{degree 1 poly}} + \underbrace{6xe^{2x}}_{\text{exponential}}$$

degree 1 poly	exponential
guess	guess
$Ax + B$	$Cxe^{2x} + De^{2x}$

Let

$$y_p = Ax + B + Cxe^{2x} + De^{2x}$$

Let's plug this into the equation.

$$y_p = Ax + B + Cxe^{2x} + De^{2x}$$

$$y_p' = A + Ce^{2x} + 2Cxe^{2x} + 2De^{2x}$$

$$y_p'' = 2Ce^{2x} + 2Ce^{2x} + 4Cxe^{2x} + 4De^{2x}$$

Plugging into $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$ gives:

$$\underbrace{\left[(4C+4D)e^{2x} + 4Cxe^{2x}\right]}_{y_p''} - 2\underbrace{\left[A + (C+2D)e^{2x} + 2Cxe^{2x}\right]}_{y_p'}$$

$$-3\underbrace{\left[Ax + B + Cxe^{2x} + De^{2x}\right]}_{y_p} = 4x - 5 + 6xe^{2x}$$

Regrouping gives:

$$-3Ax + (-2A-3B) - 3Cxe^{2x} + \underbrace{(2C-3D)}_{0}e^{2x} = 4x - 5 + 6xe^{2x}$$

We get

$$\boxed{\begin{aligned} -3A &= 4 \\ -2A - 3B &= -5 \\ -3C &= 6 \\ 2C - 3D &= 0 \end{aligned}} \Rightarrow \boxed{\begin{aligned} A &= -4/3 \\ B &= -23/9 \\ C &= -2 \\ D &= -4/3 \end{aligned}}$$

Thus,

$$y_p = -\frac{4}{3}x - \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$$

<u>Step 3:</u> The general solution to

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$$

is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{-x} + c_2 e^{3x} - \frac{4}{3}x - \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x} \end{aligned}$$

What can go wrong with the guessing?
If y_p contains terms that appear in y_h then you will have to multiply those terms by the smallest power x^n that eliminates the duplication

Let's see some examples of this

Ex: Let's solve

$$y'' - 5y' + 4y = 8e^x$$

Step 1: Solve the homogeneous equation

$$y'' - 5y' + 4y = 0$$

which has characteristic polynomial

$$r^2 - 5r + 4 = 0$$

$$(r-1)(r-4) = 0$$

So,

$$y_h = c_1 e^x + c_2 e^{4x}$$

Step 2: Let's guess a particular solution for $y'' - 5y' + 4y = 8e^x$.

Our table says to try $y_p = Ae^x$.

However plugging this in gives

$$Ae^x - 5Ae^x + 4Ae^x = 8e^x$$

$$0 = 8e^x$$

This happened because e^x is part of y_h!

What we do is we instead try

$y_p = Axe^x$ ← multiply by x

This isn't part of y_h.

We get

$$y_p = Axe^x$$
$$y_p' = Ae^x + Axe^x$$
$$y_p'' = Ae^x + Ae^x + Axe^x$$

Plugging this into $y'' - 5y' + 4y = 8e^x$ gives

$$(Ae^x + Ae^x + Axe^x) - 5(Ae^x + Axe^x) + 4Axe^x = 8e^x$$

$$-3Ae^x = 8e^x$$
$$A = -\frac{8}{3}$$

Thus, $y_p = -\frac{8}{3}xe^x$.

Step 3: The general solution to $y'' - 5y' + 4y = 8e^x$

is $y = y_h + y_p = c_1e^x + c_2e^{4x} - \frac{8}{3}xe^x$

Ex: Let's solve

$$y'' - 2y' + y = e^x$$

Step 1: Solving $y'' - 2y' + y = 0$ we have the characteristic polynomial

$$r^2 - 2r + 1 = 0$$
$$(r-1)(r-1) = 0$$

So,

$$y_h = c_1 e^x + c_2 x e^x$$

Step 2: Now to guess y_p at first we think $y_p = Ae^x$ but e^x is part of y_h. Then we try $y_p = Axe^x$ but that's also part of y_h. Thus, we try $y_p = Ax^2e^x$.

We have:

$$y_p = Ax^2e^x$$
$$y_p' = 2Axe^x + Ax^2e^x$$
$$y_p'' = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x$$

Plugging this into $y'' - 2y' + y = e^x$ we get

$$(2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x) - 2(2Axe^x + Ax^2e^x) + (Ax^2e^x) = e^x$$

Simplifying gives:

$$2Ae^x = e^x$$

Thus, $A = \frac{1}{2}$.

So, $y_p = \frac{1}{2}x^2e^x$.

<u>Step 3:</u> The general solution to $y'' - 2y' + y = e^x$ is

$$y = y_h + y_p = c_1e^x + c_2xe^x + \frac{1}{2}x^2e^x$$